By changing the independent variable from R to T by $F_1R = T$, we get

$$\frac{d^2\lambda}{dT^2} - \frac{(1+C_2)}{T}\frac{d\lambda}{dT} - \lambda + \frac{C_1}{F_1^2} = -\lambda\frac{(1+\sin^2\bar{\sigma})}{F_1^2}\frac{k^2}{2} \tag{3}$$

By neglecting the right-hand side, the solution of Eq. (3) that is obtained is more refined, as certain terms not considered in Ref. 1 are included by their average value to show their effect on the behavior of parameter λ:

$$\frac{d^2\lambda}{dT^2} - \frac{(1+C_2)}{T}\frac{d\lambda}{dT} + \frac{C_1}{F_1^2} - \lambda = 0 \tag{4}$$

By further changing the dependent variable by

$$P = (C_1/F_1^2) - \lambda$$

the differential Eq. (4) leads to

$$\frac{d^2P}{dT^2} - \frac{(1+C_2)}{T}\frac{dP}{dT} - P = 0 \tag{5}$$

The solution of Eq. (5) is now obtained in terms of the modified Bessel function of order μ.

Solution of Differential Equation

Changing the dependent variable P by the transformation

$$P = UT^{n+1/2}$$
$$n = (1+C_2)/2$$

so that U is the new dependent variable, the differential Eq. (5) becomes

$$T^2\frac{d^2U}{dT^2} + T\frac{dU}{dT} - \left[\left(\frac{2n+1}{2}\right)^2 + T^2\right]U = 0 \tag{6}$$

By writing

$$(2n+1)/2 = \mu$$

we get

$$T^2\frac{d^2U}{dT^2} + T\frac{dU}{dT} - (\mu^2+T^2)U = 0 \tag{7}$$

The differential Eq. (7) is the standard Bessel's differential equation, which admits solution as

$$U = AI_\mu(T) + BK_\mu(T) \tag{8}$$

where $I_\mu(T)$ and $K_\mu(T)$ are the modified Bessel functions of order μ and A and B are integration constants.

Hence,

$$P = T^{n+1/2}[AI_\mu(T) + BK_\mu(T)] \tag{9}$$

where

$$\mu = 1 + (C_2/2)$$

Going back to the original variables, namely, $R\sin\sigma$ and R, we obtain

$$R\sin\sigma = (C_l/F_1^2) - (F_1R)^\mu[AI_\mu(F_1R) + BK_\mu(F_1R)] \tag{10}$$

where

$$\mu = 1 + (C_2/2)$$

The constants of integration A and B can be determined with the help of conditions as used in Ref. (1).

The solution as given by Eq. (10) is more realistic when compared with that of the solution given by Eq. (48) of Ref. 1.

The solution as obtained by Lin and Tsai can be obtained easily from that given by Eq. (10) by suppressing the terms not considered by them, namely, the $F_1^2\sin^2\bar{\sigma}$ and $\sin^2\bar{\sigma}$. The solution so obtained will be the modified Bessel function of order 3/2.

Thus, the solution obtained is considered more valuable for further analysis and the determination of optimum control gains K_1 and K_2, which are expressed in terms of normal acceleration of a missile by Eq. (19) of Ref. 1. The gains K_1 and K_2 as obtained by Lin and Tsai, are modified with this solution, and their behavior is going to show the nonlinear effect in trajectory-shaping guidance.

Conclusion

In this Note, an analytical solution of the optimal trajectory-shaping guidance law has been presented. The solution offered results in better estimation of the optimal gains of a closed-loop nonlinear system. The missiles employing the above guidance law can achieve higher range with excellent intercept performances. It would also provide a smooth acceleration–time history leading to better end-performance parameters. The solution offered is in terms of well-known functions.

The analytical solution offered in Ref. 1 can be obtained directly on a particular case of this study. This guidance law finds its application for surface-to-surface, surface-to-air, and air-to-air types of missiles.

Acknowledgment

I would like to thank Dr. I Achyuta Rao, Director, Systems, Defence Research and Development Laboratory, for his kind encouragement and valuable discussions.

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Three-Dimensional Energy-State Extremals in Feedback Form

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Introduction

In Refs. 1 and 2, we have developed an algorithm for real-time near-optimal, three-dimensional guidance of high-performance aircraft in pursuit-evasion and target-interception

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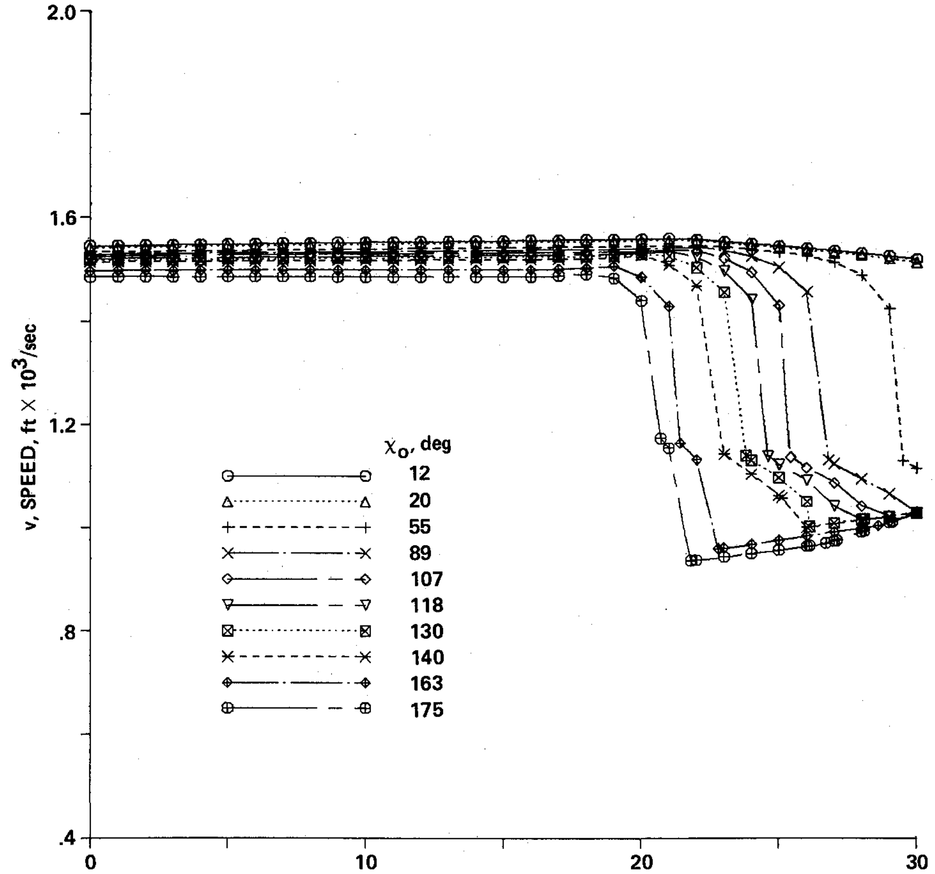
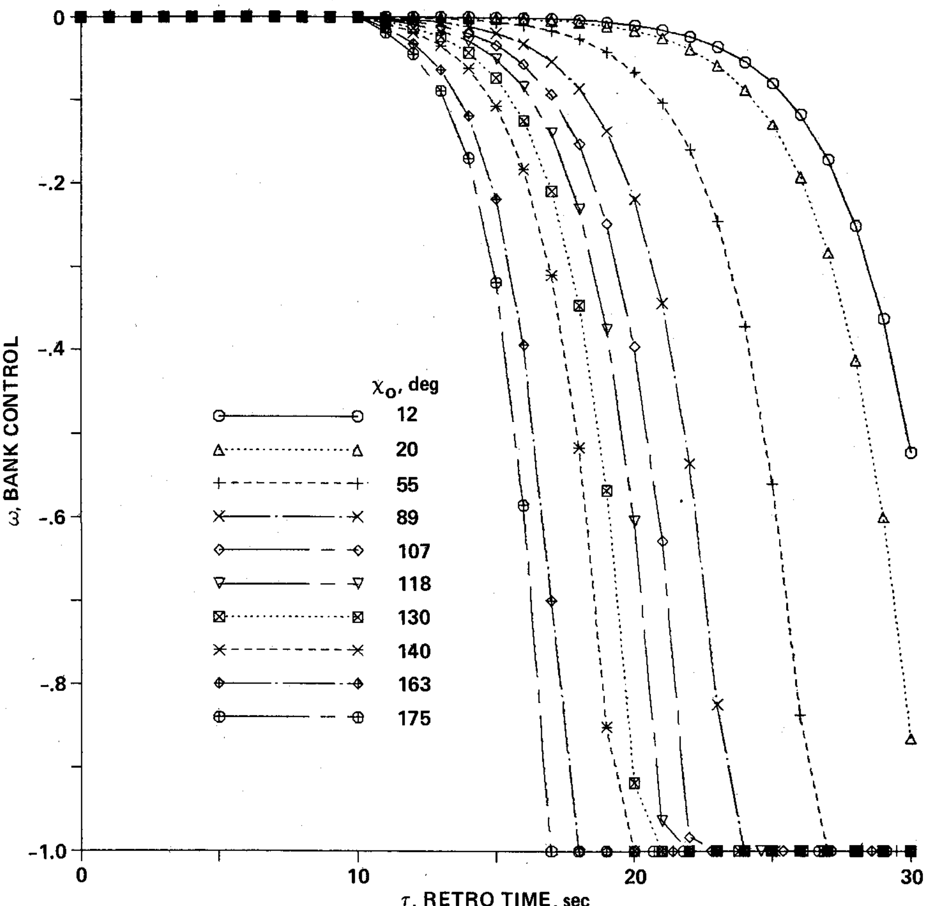
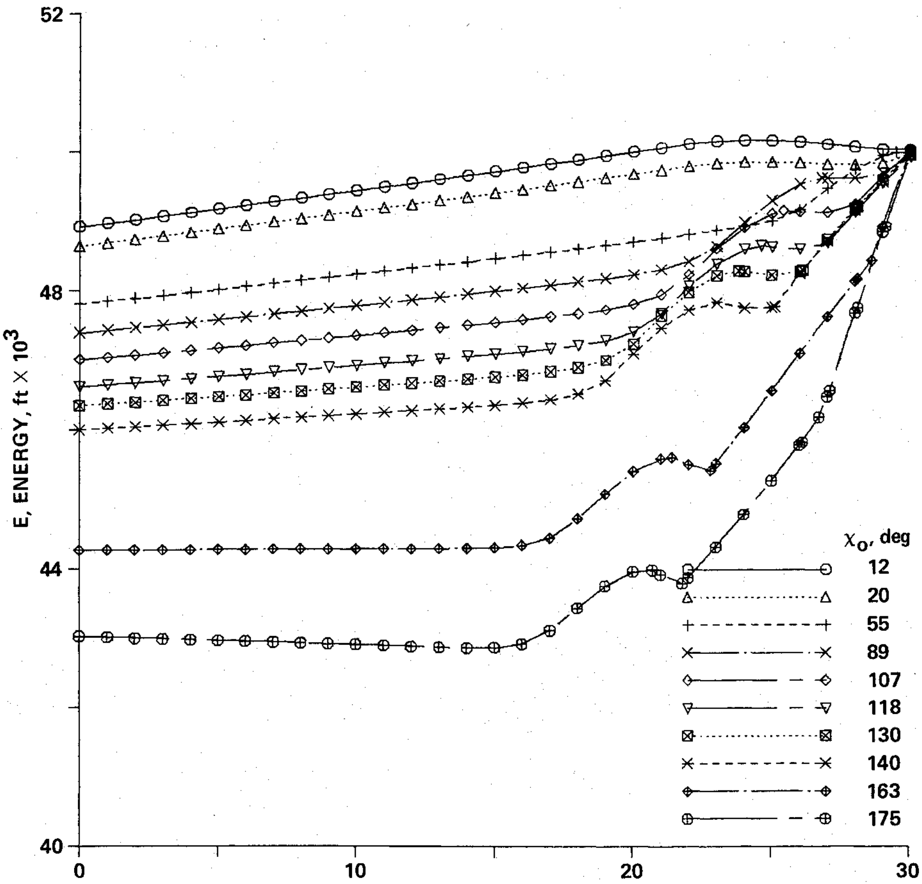


a) Speed variation

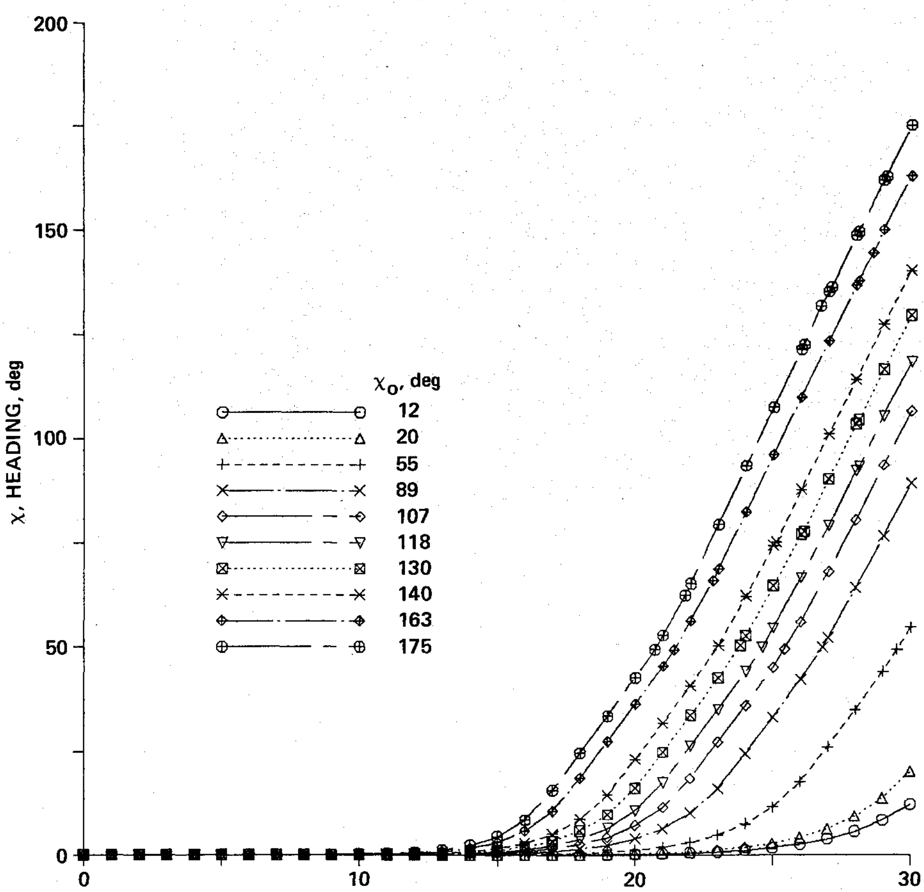


b) Bank variation

Fig. 1 Time histories of three-dimensional energy-state (3DES) extremals for F-15 aircraft for initial specific energy E_0 of 50×10^3 ft and extremal duration T of 30 s.



c) Energy variation



d) Heading variation

Fig. 1 (continued) Time histories of three-dimensional energy-state (3DES) extremals for F-15 aircraft for initial specific energy E_0 of 50×10^3 ft and extremal duration T of 30 s.

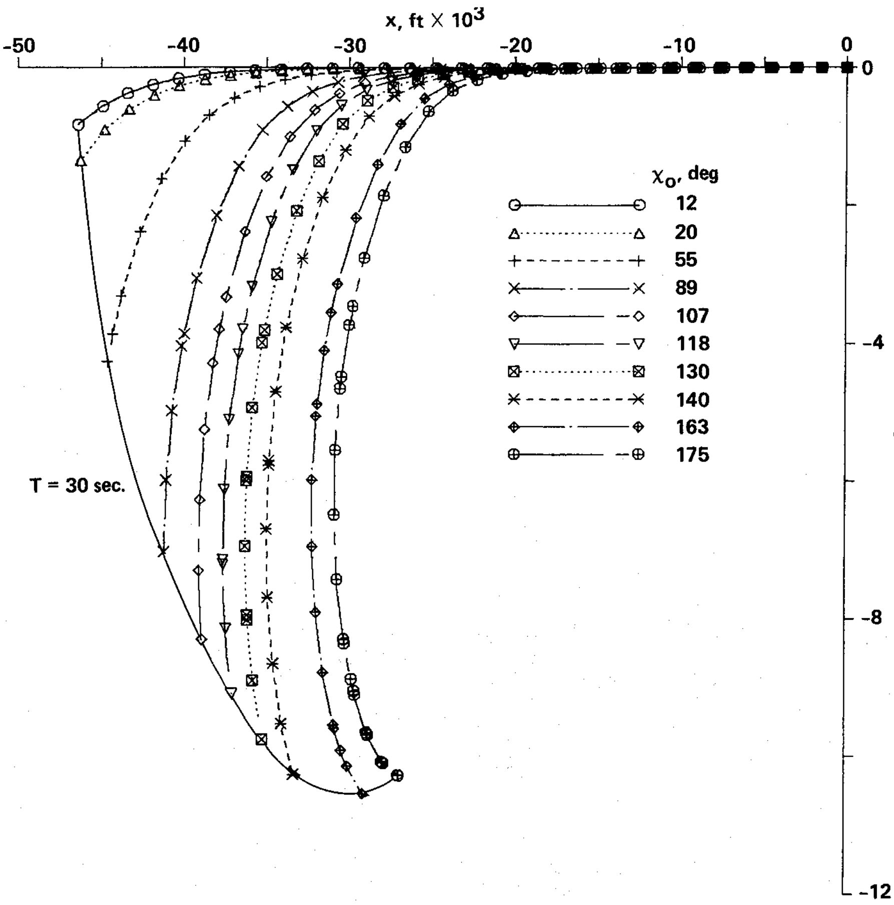


Fig. 2 Extremal (3DES) trajectory projection on a horizontal plane; F-15 aircraft, $E_0 = 50 \times 10^3$ ft, and $T = 30$ s.

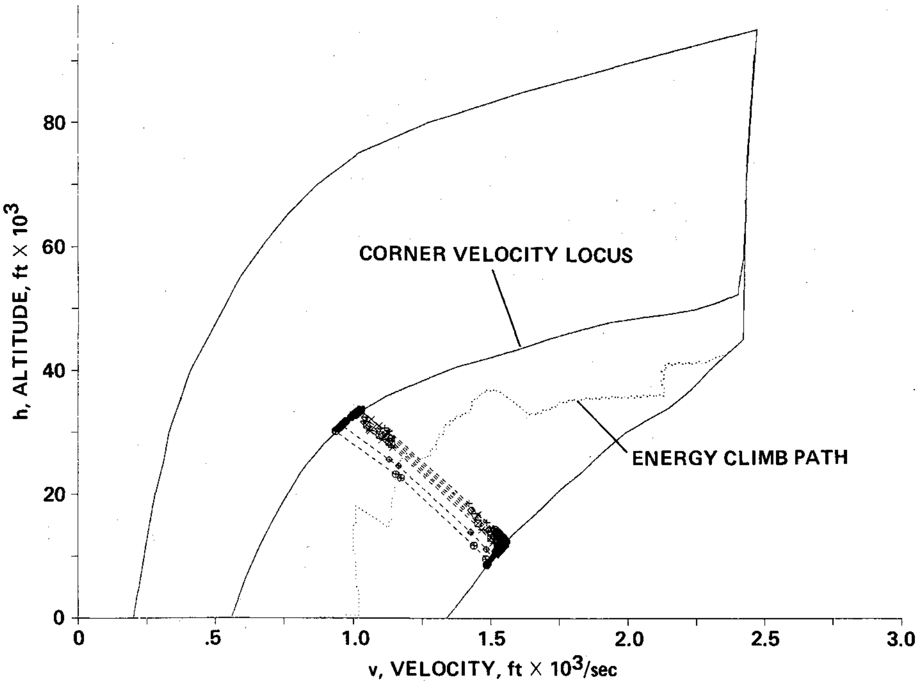


Fig. 3 Extremal (3DES) trajectories in the flight envelope; F-15 aircraft, $E_0 = 50 \times 10^3$ ft, and $T = 30$ s.

missions. The approach depends on singular perturbation methods to separate the aircraft dynamics into different time scales and on a reference frame that decouples the slow subsystem dynamics of the implementing aircraft from the motion of the opponent aircraft or target.

The time scaling adopted in Refs. 1 and 2 yields the three-dimensional energy-state (3DES) dynamic model as the reduced problem. Extremal solutions of this model in feedback form (that is, as functions of the current slow state variables) are needed for the real-time guidance law computation. In this

Note, we present and discuss example calculations of 3DES feedback extremals for a contemporary high-performance aircraft. In addition to being required for the guidance law, the 3DES extremals are of interest in their own right, both as numerical approximations to optimal trajectories and as a means of gaining insight into the nature of high-performance-aircraft optimal maneuvers.

Three-Dimensional Energy-State Extremals

The 3DES dynamic model is

$$\dot{E} = P(E, V, n, \beta), \quad \dot{\chi} = gn \sin\theta / V$$

$$\dot{x} = V \cos\chi, \quad \dot{y} = V \sin\chi, \quad n \cos\theta = 1$$

where $(\dot{\ }) = d(\ )/dt$, and the state variables are horizontal position (x,y), heading angle (χ), and specific energy (E); P is specific excess power, and n is load factor. These equations are valid in any inertial reference frame.

This dynamic model results from assuming altitude and flight-path angle to be fast variables, compared with E, χ, x, and y. Many other time-scale separations of the flight dynamics equations have been investigated; these are critically reviewed in Ref. 3. The first systematic study of three-dimensional flight-path optimization problems by singular perturbations was performed by Kelley,[4] and other noteworthy approaches may be found in Refs. 5 and 6.

In Ref. 7 we developed necessary conditions for the optimal controls β (throttle), θ (bank angle), and V (speed) for the minimum time transfer on the 3DES dynamic model and gave example extremals for a specific aircraft. These extremals are open-loop; they were produced by backward integration of the necessary conditions and are thus parameterized in terms of final conditions E_f and χ_f.

The extremals to be discussed in this Note differ in two important aspects from those of Ref. 7. First, they are closed-loop in the sense that iterative backward integrations have been performed to match specified initial conditions. Consequently, these extremals are parameterized in terms of initial conditions E_0 and χ_0, as well as trajectory duration T. Second, the extremals are computed for a current-generation aircraft, a version of the F-15, whereas those of Ref. 7 are for a version of the F-4.

Time histories of example extremal trajectories in feedback form for the F-15 aircraft are shown in Fig. 1. Two of the control variables are shown (V and ω, the latter being a ratio of bank angle to maximum available bank angle, as limited by maximum load factor and stall) and two state variables (E and χ). All trajectories have values of $E_0 = 50 \times 10^3$ ft and $T = 30$ s.

In forward time ($t = T - \tau$), Fig. 1a shows that trajectories with χ_0 greater than approximately 75 deg start on the corner velocity locus (locus of maximum instantaneous turn rate). They then make successive transitions to two higher unbounded speeds and finally end on the maximum speed bound. Trajectories with $\chi_0 < 75$ deg begin with speeds greater than the corner velocity and end at maximum speed. Figure 1b shows that trajectories with $\chi_0 > 25$ deg begin with full bank and then smoothly transition to zero bank at termination. For $\chi_0 < 25$ deg the trajectories start at less than full bank. It was also found that, for χ_0 greater than approximately 150 deg, the throttle setting started at zero and subsequently switched to full but that for $\chi_0 < 150$ deg the throttle is set at full throughout.

The paths of the extremal trajectories in the horizontal plane are shown in Fig. 2. The particular inertial reference frame chosen to display these results has its origin at trajectory termination; this coordinate system has certain advantages in pursuit-evasion.[7] (Note that the x- and y-axis scales are different.) The characteristic of hard-turning transitioning into a high-speed dash is apparent.

Figure 3 shows the trajectories in the flight envelope. Here we see clearly the behavior of trajectories starting at large heading angles: after an initial period of hard turning on the corner velocity locus, the paths transition to the vicinity of the energy climb path, where they remain briefly. They then jump to a speed near the maximum speed and finally transition to the maximum speed for the final dash. The energy climb path is the locus of most rapid energy increase in vertical flight, and the extremal control is apparently using this property to briefly accumulate energy. In fact, the period of time spent near the energy climb path coincides with the boost in energy between 20 and 25 s exhibited on Fig. 1c. The time spent on these branches of the reduced solution will, of course, be greatly reduced when boundary-layer corrections (fast dynamics) are included.

Concluding Remarks

The trajectories generated in this analysis display features that agree with aircraft pursuit-evasion and target-interception experience. They consist primarily of segments of hard turning on the corner velocity locus and high-speed dash on the maximum speed boundary, with occasional short intervals of optimal energy accumulation. The fact that these trajectories exhibit frequent large jumps between these various branches of the 3DES solution means that the transitions between these branches, the boundary-layer motions, will be important. Even in this situation, however, the reduced solution is of key importance because in singularly perturbed systems it is the reduced solution that provides the equilibrium points that are the basis of the boundary-layer computations.

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Eigenstructure of the State Matrix of Balanced Realizations

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Introduction

THE calculation of balanced representations and their properties have been studied extensively.[1-6] One aspect of these representations, however, has not been studied; namely,

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